

The Radial Part of the Zero-Mode Hamiltonian for Sigma Models with Group Target Space

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Abstract: In this note we use geometric arguments to derive a possible form for the radial part of the “zero-mode Hamiltonian” for the two dimensional sigma model with target space S^3 , or more generally a compact simply connected Lie group.

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§0. Introduction.

At the classical level a two dimensional sigma model with arbitrary Riemannian target space is conformally invariant (see §3 of [Ga], or §1 below, for the meaning of these terms). At the quantum level (assuming the theory makes sense), it is believed that conformal invariance is broken whenever the Ricci curvature of the target is nonvanishing (see §3 of [Ga]). This is the case for the targets considered in this paper, compact simply connected Lie groups with biinvariant metric.

For a given quantum 2D sigma model with compact target space, one expects that for each finite radius R , there is a Hilbert space of states, $H(S_R^1)$, associated to the circle with radius R , and there is a *trace class* operator, $U(\Sigma)$, from incoming to outgoing state spaces, associated to each oriented compact Riemannian surface Σ , with parameterized geodesic boundary components, such that sewing of surfaces corresponds to composition of operators. The sewing property expresses the locality of quantum field theory, as interpreted by Segal; the trace class condition arises from the expectation that finite numbers can be associated to the path integrals corresponding to closed surfaces (see §2.6 of [Ga]). In particular the infinitesimal generator for the one parameter family $U(S_R^1 \times [0, t])$, the Hamiltonian H_R , should have a discrete spectrum.

In the case of a conformally invariant quantum sigma model, the Hilbert space $H(S_R^1)$ and Hamiltonian H_R are essentially independent of R . Otherwise there is dependence on R , and for the models we consider, it is expected that in the limit $R \uparrow \infty$, the Hamiltonian $H = H_\infty$ has a mass gap and continuous spectrum.

In this paper we will assume that the Hilbert spaces $H(S_R^1)$ have a certain concrete mathematical form. We will also introduce some simplifying assumptions regarding the

action of the Hamiltonian operators H_R . On the basis of these assumptions, we will draw some conclusions about the form of the radial part of the “zero-modes” of these operators.

The arguments in this paper are most complete in the limiting case $R \uparrow \infty$. We assume that in this limit the space of states of the 2D sigma model, with simply connected group target space K , has the form $\mathcal{H}(S_\infty^1) = L^2(\mu)$, where μ is a certain canonical measure on a distribution-like completion of the loop space LK . We think of this measure as specifying the vacuum for the theory. This is motivated by the known form for the vacuum of the conformally invariant WZW model (see §4.1 of [Ga]). We also assume that the Hamiltonian acts as a second order differential operator on a certain subspace of zero-mode states. In the case of S^3 , on the basis of these assumptions, we find that the radial part of this “zero-mode Hamiltonian” is equivalent to

$$-(\frac{d}{dr})^2 + \frac{1}{4} - \frac{15}{4} \text{sech}^2(r), \quad (0.1)$$

(acting on odd functions) up to a scale factor (the mass parameter). This operator has a unique ground state and a mass gap. The spectrum of this radial part does not reflect expected features of the full spectrum, such as jumps in multiplicity corresponding to multiparticle states. There is a mechanism which should produce these jumps in the spectrum of the full zero-mode Hamiltonian, namely the discreteness of the $K \times K$ isotypic decomposition. Unfortunately the zero-mode Hamiltonian cannot be determined on mathematical grounds alone from its radial part.

In §1 I will more fully explain the motivation for the conjecture. This involves a brief review of the Hamiltonian formalism for the two dimensional sigma model.

In §2 I will briefly review some of the mathematics which §1 depends upon, especially the construction of the measure μ (which is carried out in [Pi]). This involves understanding the limit of Wiener measure on LK , in terms of Riemann-Hilbert factorization, as inverse temperature tends to zero (which corresponds to $R \rightarrow \infty$). For the purposes of this paper, the key is Step 7 in §2, which is a conjectured formula for the spherical transform of the diagonal (or zero-mode) distribution of the measure μ , in terms of an affine analogue of Harish-Chandra’s famous \mathbf{c} -function (the mathematical conclusions of this paper depend exclusively on this affine \mathbf{c} -function, and many readers may care to skip the infinite dimensional measure-theoretic considerations; however, it should be borne in mind that the fact this function arises as a transform for a measure on (distributional) loop space is a key argument for the plausibility of our physical claims).

In §3 I will present the calculations which lead to (0.1), and to an analogous conjecture for the massive deformation of the conformally invariant WZW model.

In §4 I will discuss the case of finite radius. The mathematical underpinnings for this section are not as strong as in the case $R \uparrow \infty$, because the analogue of the measure μ has not been rigorously constructed. However there does appear to be a natural conjecture for the zero-mode distribution. This involves understanding a quantum (theta function) deformation of the affine \mathbf{c} -function. Most of this section revolves around some nontrivial positivity checks for this deformation.

§1. Origins of the Conjecture.

We initially suppose that space is the circle, S^1 , so that spacetime is $\Sigma = S^1 \times \mathbb{R}$, with coordinates (θ, t) . We also initially suppose that the target space is X , an arbitrary Riemannian manifold.

The classical fields for the sigma model with target space X are maps $x : \Sigma \rightarrow X$, and the action is the kinetic energy function

$$\mathcal{A}(x) = \frac{1}{2} \int_{\Sigma} \langle dx \wedge *dx \rangle = \frac{1}{2} \int_{\Sigma} \{ |\frac{\partial x}{\partial t}|^2 + |\frac{\partial x}{\partial \theta}|^2 \} d\theta dt. \quad (1.1)$$

This action is conformally invariant, meaning that if the metric ds of Σ is changed to ρds , where ρ is a positive function, the action remains unchanged. In particular the action depends upon the radius of the circle and time scale in a covariant way.

The time zero fields constitute the loop space $LX = \text{Map}(S^1, X)$ (when it is useful to denote the degree of smoothness, we will use a subscript, e.g. $L_{C^0}X$ will denote the manifold of continuous loops). The tangent space to LX at x is naturally identified with $\Omega^0(x^*TX)$, the space of vector fields along the loop x . There is a Riemannian metric on this tangent space, given by

$$\langle v, w \rangle_x = \int_{S^1} \langle v(\theta), w(\theta) \rangle_{x(\theta)} d\theta, \quad (1.2)$$

where $v(\theta), w(\theta) \in TX|_{x(\theta)}$, and $\langle \cdot, \cdot \rangle_{x(\theta)}$ denotes the inner product (Riemannian metric) for X at the point $x(\theta)$. In this way we can view $L_{C^0}X$ as a Riemannian manifold.

In the second expression in (1.1) for \mathcal{A} , the first term is the usual kinetic energy for a path in the Riemannian manifold $L_{C^0}X$, and the second term represents a potential energy term, corresponding to the energy function on the finite energy loop space $L_{W^1}X$,

$$E(x : S^1 \rightarrow X) = \frac{1}{2} \int_{S^1} \langle dx \wedge *dx \rangle = \frac{1}{2} \int |\frac{\partial x}{\partial \theta}|^2 d\theta. \quad (1.3)$$

Note that the Riemannian metric (1.2) and E depend upon the radius of S^1 .

From (1.1) – (1.3) we can deduce, in a rough heuristic way, that the quantum Hamiltonian for the sigma model is of the form

$$H = \Delta + E \quad (1.4)$$

where Δ is the Laplacian for the Riemannian manifold $L_{C^0}X$, and E is viewed as a (extremely singular) multiplication operator. At this heuristic level, the operator H should define a nonnegative self-adjoint operator on a Hilbert space that is of the form “ $L^2(LX, dV)$ ”, where dV is a fictional Riemannian volume element.

To rigorously define H , one must introduce a cutoff which breaks scale covariance, and (as discussed in the introduction) it is expected that in general, after removing the cutoff, there is a residual nontrivial dependence of H on R , the radius of the circle.

One can at least schematically think of one aspect of this renormalization process in terms of a commutative diagram (involving unbounded operators)

$$\begin{array}{ccc} L^2(\Omega_R^2 dV) & \xrightarrow{\Omega \circ (\Delta + E) \circ \Omega^{-1}} & L^2(\Omega_R^2 dV) \\ \uparrow \Omega^{-1} & & \uparrow \Omega^{-1} \\ L^2(dV) & \xrightarrow{\Delta + E} & L^2(dV) \end{array}, \quad (1.5)$$

where $\Omega = \Omega_R$, the fictional ground state for $\Delta + E$ (depending upon the radius R of the circle), is viewed as a multiplication operator. In this diagram the coupled pair $\Omega_R^2 dV$ should represent a well-defined finite positive measure (when topological terms are added to the action, this might more generally represent a measure having values in a line bundle). The point is that to obtain a well-defined Hamiltonian, one must consider states relative to the ground state.

Example (see §1.3 of [Ga]). Suppose $X = \mathbb{R}$, and for simplicity, we add an explicit mass term $m^2|x(t, \theta)|^2$ to the integrand in (1.1). By expanding $x(t, \theta) = \sum x_k(t)e^{ik\theta}$ in a Fourier series, one sees that $\Delta + E$ can be written as a sum of oscillators. The formal Hilbert space $L^2(L\mathbb{R}, dx)$ and the ground state

$$\Omega = \exp\left(-\frac{1}{2} \sum_{k=-\infty}^{\infty} \sqrt{m^2 + k^2} |x_k|^2\right) \quad (1.6)$$

do not make sense individually. But in the top row of the diagram

$$\begin{array}{ccc} L^2\left(\frac{1}{\mathcal{Z}} e^{-\sum \sqrt{m^2 + k^2} |x_k|^2} dx\right) & \xrightarrow{\Omega \circ (\Delta + E) \circ \Omega^{-1}} & L^2\left(\frac{1}{\mathcal{Z}} e^{-\sum \sqrt{m^2 + k^2} |x_k|^2} dx\right) \\ \uparrow \Omega^{-1} & & \uparrow \Omega^{-1} \\ L^2(L\mathbb{R}, dx) & \xrightarrow{\Delta + E} & L^2(L\mathbb{R}, dx) \end{array} \quad (1.7)$$

the measure is a well-defined Gaussian measure, and the operator can be rigorously defined.

Before discussing what is mathematically known about operators on LX (when X is curved), we digress to recall an idea inspired by Witten's work. Suppose that Y is a finite dimensional Riemannian manifold (but we will want to heuristically apply this to $Y = LX$), and that \mathcal{E} is a (Morse) function on Y . There is a commutative diagram (involving unbounded operators)

$$\begin{array}{ccc} L^2(Y, e^{-\beta \mathcal{E}} dV) & \xrightarrow{\Delta_\beta} & L^2(Y, e^{-\beta \mathcal{E}} dV) \\ \uparrow e^{\frac{\beta}{2} \mathcal{E}} & & \uparrow e^{\frac{\beta}{2} \mathcal{E}} \\ L^2(Y, dV) & \xrightarrow{\Delta_Y} & L^2(Y, dV) \end{array}, \quad (1.8)$$

akin to (1.5), where $\Delta_\beta = e^{\frac{\beta}{2} \mathcal{E}} \circ \Delta_Y \circ e^{-\frac{\beta}{2} \mathcal{E}}$, and dV denotes the Riemannian volume element (a refinement of this, involving the conjugation of d , the exterior derivative, is relevant to Morse theory and the supersymmetric sigma model; see [Wil]).

In the case $Y = L_{W^1}X$, there is a natural choice for \mathcal{E} , $\mathcal{E} = E$, there is an analogue of $e^{-\beta\mathcal{E}}dV$, namely Wiener measure with inverse temperature β , denoted ν_β (although note this measure is not supported on Y), and there is an analogue of Δ_β , which has been investigated by Gross and others. From the sigma model point of view, one can view these objects as regularizations of the heuristic expressions that we introduced above. This motivates the study of possible limits of Wiener measure ν_β , and Δ_β , plus a potential function that might stand in for E , as $\beta \downarrow 0$ (but note that Δ_β by itself tends to the Laplacian for the W^1 loop space, not the C^0 loop space). Note also that letting $\beta \downarrow 0$ corresponds to $R \uparrow \infty$.

Now suppose that $X = K$, a compact simply connected Lie group with a simple Lie algebra, \mathfrak{k} . This space has an essentially unique biinvariant Riemannian structure, determined by a multiple of the Killing form, which we will normalize in a standard way (the length squared of a long root is 2; in the case $K = SU(2, \mathbb{C}) = S^3$, this means the inner product is $\langle x, y \rangle = \text{tr}_{\mathbb{C}^2}(x^*y)$, for $x, y \in \mathfrak{su}(2, \mathbb{C})$). Eliminating this normalization would introduce a mass parameter.

A first fact of note is that Gross has proven that for a large class of potentials, $\{V\}$, $\Delta_\beta + V$ has a unique ground state (see [Gr]; the nature of the spectrum apparently remains unknown). I am unaware of any results concerning limits of these operators as $\beta \downarrow 0$, which in the present context amounts to removing a regularization. Our speculations to follow are possibly related to these limits (and hence to an infinite radius limit).

Let G denote the complexification of K (if $K = SU(2)$, then $G = SL(2, \mathbb{C})$). As I will describe in §2, there is a natural completion of the loop space LG , the hyperfunction loop space $L_{hyp}G$, with the properties that (1) LK acts from the left and right, and (2) the Wiener measures ν_β converge to a biinvariant probability measure μ on $L_{hyp}G$ as $\beta \downarrow 0$. This depends in an essential way on our assumption that K is simply connected. The measure μ should be characterized in the following way:

(1.9) Conjecture. *There is a unique probability measure on $L_{hyp}G$ which is biinvariant with respect to LK (for uniqueness it should suffice to consider polynomial loops).*

In this paper we assume that the Hilbert space for the sigma model with target space K , in the infinite radius limit, is $L^2(L_{hyp}G, \mu)$, and we propose to use the structure of $L_{hyp}G$ and μ to infer properties of the Hamiltonian $H = H_\infty$.

(1.10) Remarks. (a) We emphasize that μ is a probability measure. Even in a heuristic sense, μ is not to be confused with the fictional Riemannian volume element dV on $L_{C^0}K$. We think of switching from dV to μ as similar to the vacuum renormalization process in (1.5), with $R = \infty$.

(b) In §4 we will discuss the $R < \infty$ case, as best we understand it. At this time it is not clear how to formulate a characterization of the corresponding measure, $\Omega_R^2 dV$ (assuming it exists), similar to (1.9).

A generic $g \in L_{hyp}G$ can be represented as a formal product

$$g = g_- \cdot g_0 \cdot g_+, \tag{1.11}$$

where $g_0 \in G$ is constant, and g_{\pm} are G -valued holomorphic functions on the disks $\Delta = \{|z| < 1\}$ and $\Delta^* = \{|z| > 1\}$, respectively, with $g_+(0) = 1$ and $g_-(\infty) = 1$. If $g \in L_{C^0}G$ (an ordinary continuous loop in G) and generic (i.e. the Toeplitz operator associated to g is invertible), then (1.11) is the standard triangular or Riemann-Hilbert or Birkhoff factorization of g (see [CG] or chapter 8 of [PS]).

There is a strongly motivated conjecture for the g_0 distribution of μ ; in the case $K = SU(2)$, the conjecture states that

$$(g_0)_*\mu = \frac{1}{\mathcal{Z}} \text{tr}(g_0^* g_0)^{-3} dm(g_0), \quad (1.12)$$

where dm denotes an invariant measure for $SL(2, \mathbb{C})$, and \mathcal{Z} normalizes the total mass to be one (see (3.4) below for the general formula).

We now introduce two further assumptions. The first is that the low energy states of the sigma model should be functions of g_0 alone. The second is that the Hamiltonian, or an approximation to it, should act on the space $L^2(G, (g_0)_*\mu)$, and that this approximation should be given by a second order operator, necessarily biinvariant with respect to K . We will refer to this approximation as the “zero-mode Hamiltonian”, since functions of g_0 are rotation invariant (but note that $L^2(G, (g_0)_*\mu)$ is properly contained in the space of all rotation invariant functions!). In the case of $K = SU(2)$, K biinvariance leaves just one radial degree of freedom for the radial part, and as we will calculate in §3, this leads to (0.1).

We will now briefly indicate how this generalizes to include other action terms.

Returning to a general target space X , given a “ B -field” on X (i.e. an element $b \in \hat{H}^2(X, \mathbb{T})$, the degree 2 Cheeger-Simons differential characters, which can be written heuristically as $b = \exp(2\pi i B)$, where B is a 2-form on X), there is a multivalued generalization of the sigma model action, which gives rise to well-defined Feynmann amplitudes,

$$\exp(-\beta \mathcal{A}(x) + 2\pi i \int_{\Sigma} x^* B). \quad (1.13)$$

The deformation invariant of a B -field is the cohomology class of dB in $H^3(X, \mathbb{Z})$.

In the case $X = K$, there are special B -fields, the WZW action terms, which are parameterized by a level $l \in \mathbb{Z} = H^3(K, \mathbb{Z})$. When the inverse temperature parameter β and the level l satisfy $\beta = l$, then the corresponding sigma model is the conformally invariant WZW model at level l , for which the Hilbert space is

$$H_{L^2}^0(\mathcal{L}^{*\otimes l}), \quad (1.14)$$

the space of holomorphic sections of the line bundle $\mathcal{L}^{*\otimes l}$, where $\mathcal{L} = \hat{L}_{hyp}G \times_{\mathbb{C}^*} \mathbb{C}$, and the vacuum is (an appropriate power of) the Toeplitz determinant, $\det A(\hat{g})$, viewed as a section (it is worth noting that in this exceptional example, vacuum renormalization, as in (1.5), is not necessary). It is well-known that this space decomposes discretely into

irreducible representations with respect the action of $\hat{L}K \times \hat{L}K$ (the Kac-Moody extensions of the loop groups), and one can use this to find the (discrete) spectrum of the model. In this conformally invariant context the various assumptions we made above, about the dependence of low energy states on g_0 and so on, are, in some sense, known to be correct (see [Ga]).

We consider the ansatz that the Hilbert space for the corresponding massive deformation at level l is the larger space of all sections,

$$\Omega_{L^2}^0(\mathcal{L}^{*\otimes l}). \quad (1.15)$$

How the orthogonal complement of the discrete part (1.14) decomposes, if at all, is simply not known. However it is again reasonable to investigate the possibility that in terms of the Riemann-Hilbert factorization (1.11), the low energy states depend only upon g_0 , and so on. We will write down the conjectural radial part of the zero-mode Hamiltonian at level l in §3.

§2. The Structure of μ .

The existence of biinvariant limits of the measures ν_β as $\beta \downarrow 0$ is proven in [Pi]. Here I will give an outline of a relatively direct proof. The argument is broken into seven steps, two of which are listed as conjectural. Conjectural step 5 can be dispensed with; the results of [Pi] can be used to bypass this step, but this detour is long and step 5 is of considerable intrinsic interest. Conjectural step 7, which gives an explicit formula for the g_0 distribution of μ , is essential for the purposes of this paper.

By definition (see chapter 2, Part III, of [Pi]), as a set,

$$L_{hyp}G = G(\mathcal{O}(S^{1-})) \times_{G(\mathcal{O}(S^1))} G(\mathcal{O}(S^{1+})), \quad (2.1)$$

where $G(\mathcal{O}(S^1))$ is the group of analytic loops in G , $G(\mathcal{O}(S^{1+}))$ is the direct limit of the groups $G(\mathcal{O}(\{r < |z| < 1\}))$ as $r \uparrow 1$ (G -valued holomorphic functions on some annulus just inside S^1), $G(\mathcal{O}(S^{1-}))$ is the direct limit of the $G(\mathcal{O}(\{1 < |z| < r\}))$ as $r \downarrow 1$, and $G(\mathcal{O}(S^1))$ acts on these latter two groups by multiplication. This is a nonabelian generalization of Sato's realization of the dual of $\mathcal{O}(S^1)$ (the elements of this dual are called hyperfunctions, and generalize the notion of a distribution). From this global definition it is clear that $G(\mathcal{O}(S^1))$ acts on the left and right of $L_{hyp}G$ (but this action is far from transitive). The set $L_{hyp}G$ can be turned into a complex manifold, where a model coordinate neighborhood is given by (1.11); the coordinates for this neighborhood are

$$(\theta_-, g_0, \theta_+) \in H^1(\Delta^*, \mathfrak{g}) \times G \times H^1(\Delta, \mathfrak{g}) \quad (2.2)$$

where $\theta_+ = g_+^{-1} \partial g_+$ and $\theta_- = (\partial g_-) g_-^{-1}$. Other neighborhoods are obtained by translation by elements of $G(\mathcal{O}(S^1))$.

(2.3) Technical Remark. Below it will occasionally be useful to replace θ_+ by its integral $x_+ \in H^0(\Delta, \mathfrak{g})_0$, where $\theta_+ = \partial x_+$, $x_+(0) = 0$. One could imagine using other coordinates as well. But (2.2) is natural in the following sense: there is a natural action of $\text{Diff}_{C^\omega}^+(S^1)$ on $L_{hyp}G$, in addition to the action of $LK \times LK$; the coordinates (2.2) are equivariant with respect to the subgroup $PSU(1,1)$, where $PSU(1,1)$ acts naturally on $H^1(\Delta, \mathfrak{g})$. Assuming the truth of our conjecture (1.9), the measure μ is invariant with respect to these actions.

There is a natural inclusion of $L_{C^0}G \rightarrow L_{hyp}G$; this follows from the existence of Riemann-Hilbert factorization for continuous loops. Wiener measure ν_β on $L_{C^0}K$ can therefore be viewed as a probability measure on $L_{hyp}G$. We recall that ν_β is characterized in the following way: given vertices $\{v\}$ and associated edges $\{e\}$ around S^1 , the distribution of the values $\{g(v)\}$ is given by the probability measure on $\prod_{\{v\}} K$:

$$\frac{1}{Z} \prod_{\{e\}} p_{Tl(e)}(g_{\partial e}) \prod_{\{v\}} dg_v, \quad (2.4)$$

where $T = 1/\beta$, p_t denotes the heat kernel for K [in particular $p_t(g, h) \sim \frac{1}{Z} \exp(-\frac{1}{2t}d(g, h)^2)$ as $d(g, h) \rightarrow 0$], and $g_{\partial e}$ denotes the pair of values of g at the ends of the edge e . Unfortunately this characterization is not directly useful in understanding ν_β in terms of the Riemann-Hilbert coordinates θ_-, g_0, θ_+ .

We now turn to the basic steps of the argument.

Step 1. ν_β is quasiinvariant with respect to $L_{W^1}K$ (finite energy loops) acting on $L_{C^0}K$ from either the left or right.

Step 2. ν_β is asymptotically invariant as $\beta \downarrow 0$ in the following precise sense: for each $p < \infty$, given $g' \in L_{W^1}K$,

$$\int_{LK} \left| \frac{d\nu_\beta(g')}{d\nu_\beta(g)} - 1 \right|^p d\nu_\beta(g) \leq 2c(\beta) \Gamma\left(\frac{p+1}{2}\right) (2\beta E(g'))^{p/2},$$

where $c(\beta) \rightarrow 1$ as $\beta \rightarrow 0$. There is a similar estimate for g' acting on the right.

Step 3. With ν_β probability one, g has a Riemann-Hilbert factorization as in (1.11), and g_\pm and x_\pm have the same “smoothness properties” as g , where $\partial x_+ = g_+^{-1} \partial g_+$, $x_+(0) = 0$.

“Smoothness” in Step 3 can be understood in various ways. A version sufficient for our purposes is the following. It is known that with ν_β probability one, g has a derivative of order s in a Sobolev (or Holder) sense, for any $s < 1/2$. According to Step 3, the same is true for g_\pm and x_\pm . In particular we have

$$\sum_{n>0} n^\alpha |\hat{x}_+(n)|^2 < \infty, \quad a.e. \quad [\nu_\beta], \quad (2.5)$$

for each $\alpha < 1$.

These first three steps are true for an arbitrary compact type Lie group K . In particular the first two steps involve a reduction to a linear situation via the use of stochastic analysis (see §4.1 of Part II of [Pi]). The third step depends fundamentally on the fact that the conjugation operator is continuous on the class of Holder continuous functions, C^μ , for any $0 < \mu < 1$ (see 2. on p 60 of [CG]).

The next step depends crucially upon the simple connectedness of K . In the case $K = SU(2)$ case, if we write

$$g_+(z) = 1 + \begin{pmatrix} a_1(g) & b_1(g) \\ c_1(g) & -a_1(g) \end{pmatrix} z + \hat{g}_+(2)z^2 + \dots \quad (2.6)$$

a straightforward calculation shows that for $k = \begin{pmatrix} a & bz \\ -\bar{b}z^{-1} & \bar{a} \end{pmatrix} \in SU_2^\tau \subset LSU_2$,

$$b_1(gk^{-1}) = \frac{ab_1(g) - b}{\bar{b}b_1(g) + \bar{a}}. \quad (2.7)$$

In other words, the right action of $k \in SU_2^\tau$ on $g \in L_{hyp}G$ intertwines with the natural linear fractional action of SU_2 on $b_1 \in \hat{\mathbb{C}}$ (SU_2^τ is a subgroup of LSU_2 which is conjugate to SU_2 via an outer automorphism τ , hence the notation). This latter action is transitive and completely determines the form of an invariant measure. Since the ν_β are asymptotically invariant, the b_1 distributions are asymptotically invariant with respect to this SU_2 action on $\hat{\mathbb{C}}$. This leads to the following conclusion.

Step 4. *In the limit as $\beta \rightarrow 0$, the distribution of b_1 is the SU_2 -invariant distribution on $\hat{\mathbb{C}}$,*

$$\frac{1}{\mathcal{Z}}(1 + |b_1|^2)^{-2} dm(b_1).$$

The behavior of the measures $(b_1)_*\nu_\beta$ contrasts sharply with the behavior of the Gaussian measures $\frac{1}{\mathcal{Z}}\exp(-\beta x^2)dx$ on Euclidean space (which is what we encounter for $K = T$, a flat torus), as $\beta \rightarrow 0$, because the “probabilistic mass” of the latter measures escapes to ∞ as $\beta \rightarrow 0$. One theme of this note is that the preservation of probabilistic mass, which depends essentially on the semisimplicity of K , is related to the existence of a mass gap for the sigma model.

(2.8) Remarks. (a) For a general simply connected K , there is a result similar to step 4, where b_1 is replaced by the coordinate for the highest root space of \mathfrak{g} .

(b) Note that $\theta_0 = \hat{\theta}_+(0) = \hat{x}_+(1) = \hat{g}_+(1) \in \mathfrak{g}$. Conjecturally,

$$\lim_{\beta \rightarrow 0} (\theta_0)_*\nu_\beta = \frac{1}{\mathcal{Z}}(1 + |\theta_0|^2)^{-d-1} dm(\theta_0), \quad (2.9)$$

where $d = \dim_{\mathbb{C}}(\mathfrak{g})$. But at this point there does not exist even a conjectural explicit formula for the joint distribution of all the modes $\theta_0, \theta_1 = \hat{\theta}_+(1), \dots$

Using invariance we can now use Steps 3 and 4 to show that the distributions of all the coefficients for g_+ (or x_+ or θ_+) assume a finite shape as $\beta \rightarrow 0$. This is proven in [Pi], using an induction argument. I believe, however, that there is a more elegant explanation, possibly useful in a more general context. A corollary of (2.4) is that for fixed $\alpha < 1$, and for each $R > 0$,

$$\nu_\beta\{n^\alpha|\hat{x}_+(n)|^2 > R\} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.10)$$

It is natural to ask whether there exists α such that the sequence in (2.10) is actually nonincreasing (but not necessarily going to 0), for all β and R ; if so there exists a largest such α , α_c . In the abelian case one can easily calculate that $\alpha_c = 2$.

Conjectural Step 5. *For $\alpha = 0$, (2.5) is a nonincreasing function of n , for all $\beta > 0$ and $R \geq 0$, i.e. $\alpha_c \geq 0$. In particular*

$$\nu_\beta\{|\hat{x}_\pm(n)| > R\} \leq \nu_\beta\{|\hat{x}_\pm(1)| > R\}.$$

The following is a consequence of Step 4 and (conjectural) Step 5.

Step 6. *There exists a constant d (depending only upon \mathfrak{g}) such that*

$$\lim_{\beta \rightarrow 0} \nu_\beta\{|\hat{x}_\pm(n)| > R\} \leq \frac{d}{(1 + (R/d)^2)}. \quad (2.12)$$

This step implies that the mass of the ν_β does not escape to ∞ as $\beta \rightarrow 0$, at least when we consider the θ_\pm coordinates. This has already been done in [Pi] in a qualitative way; the point of (2.12) is to quantify this result in an elegant way. To complete our outline, we need to know that mass does not escape to infinity through g_0 . Again, this has already been done in a qualitative way in [Pi], but we need an explicit formula.

Suppose that we choose a maximal torus T for K , and a choice of positive roots for the action of the corresponding Cartan subalgebra \mathfrak{h} of \mathfrak{g} . We can generically write $g_0 \in G$ in triangular form, $g_0 = l_0 m \mathbf{a} u_0$, where we have further decomposed the diagonal term into a phase $m \in T$ and its magnitude $\mathbf{a} \in \exp(\mathfrak{h}_\mathbb{R})$.

Conjectural Step 7. *We have*

$$\lim_{\beta \downarrow 0} \int \mathbf{a}(g)^{-i\lambda} d\nu_\beta^{**}(g) = \prod_{\alpha > 0} \frac{\sin(\frac{\pi}{2\dot{g}} \langle \rho, \alpha \rangle)}{\sin(\frac{\pi}{2\dot{g}} \langle \rho - i\lambda, \alpha \rangle)} \quad (2.13)$$

where \dot{g} is the dual Coxeter number, ρ is the sum of the positive roots, and $\lambda \in \mathfrak{h}_\mathbb{R}^*$ (and recall that the inner product has been normalized).

In the case of $K = SU_2$, this is equivalent to (1.12). The original motivation for this conjecture is explained in §4.4 of Part III of [Pi]. This formula should be compared with the known formula of Harish-Chandra,

$$\lim_{\beta \uparrow \infty} \int \mathbf{a}(g)^{-i\lambda} d\nu_\beta(g) = \mathbf{c}(\rho - i\lambda) = \prod_{\alpha > 0} \frac{\langle \rho, \alpha \rangle}{\langle \rho - i\lambda, \alpha \rangle} \quad (2.14)$$

(see §4.4 of Part II of [Pi]).

When we incorporate the level l , the generalization of conjectural step 8 is

$$\lim_{\beta \downarrow 0} \int \mathbf{a}^{-i\lambda} d\nu_{\beta,l} = \prod_{\alpha > 0} \frac{\sin(\frac{\pi}{2(\dot{g}+l)} \langle \rho, \alpha \rangle)}{\sin(\frac{\pi}{2(\dot{g}+l)} \langle \rho - i\lambda, \alpha \rangle)} \quad (2.15)$$

As $l \rightarrow \infty$, we recover the classical limit of Haar measure, (2.14). If we write $(g_0)_* \mu_l = \phi_l dm(g_0)$, then (2.15) is equivalent to the following formula for the Harish-Chandra transform:

$$(\mathcal{H}\phi_l)(\lambda) = c \prod_{\alpha > 0} \frac{\langle -i\lambda, \alpha \rangle}{\sin(\frac{\pi}{2(\dot{g}+l)} \langle -i\lambda, \alpha \rangle)} = \prod_{\alpha} \Gamma(1 + i \frac{\pi \langle \lambda, \alpha \rangle}{2(\dot{g}+l)}) \quad (2.16)$$

(this follows from (4.4.27) of Part II of [Pi]).

Steps 6 and 7 imply that the measures ν_{β} have limits in $L_{hyp}G$ as $\beta \rightarrow 0$. Asymptotic invariance implies that these limits are biinvariant with respect to analytic loops in K . The remaining step is to show that there is a unique such measure. Considerable progress has been made, but this question remains open.

(2.17)Remark. Although not directly relevant in this paper, we mention that there are conjectural expressions for the θ_{\pm} distributions, at least in terms of other, more explicit, limits. For example, for $K = SU(n)$ in the defining representation, conjecturally

$$(\theta_-)_* \mu_l = \lim_{n \rightarrow \infty} \frac{1}{Z} \det(1 + Z^* Z)^{-2-l} dm(P_n \theta_-), \quad (2.18)$$

where $Z = Z(g_-) = C(g_-)A(g_-)^{-1}$ (following the notation in [PS]), g_- corresponds to $P_n \theta_-$, and P_n projects θ_- to its first n coefficients (so that it is an orthogonal projection for $H^1(\Delta^*, \mathfrak{g})$). This expression is manifestly $PSU(1, 1)$ invariant. This is the analogue of a well-known formula of Harish-Chandra for the invariant measure on a finite dimensional flag space (see [Helg], Thm 5.20, p 198).

§3. The Conjecture for the Radial Part ($R = \infty$).

We introduce the ansatz that the subspace

$$L^2(G, (g_0)_* \mu) \subset L^2(\mu) \quad (3.1)$$

is invariant, or at least approximately invariant, with respect to the action of the Hamiltonian, $H = H_{\infty}$. This approximation, H_G , will necessarily be a $K \times K$ -invariant linear operator. To further restrict the possibilities, we also assume that H_G is a second order differential operator.

Consider the Cartan decomposition

$$\psi : K \times \mathfrak{p} \rightarrow G : k, x \rightarrow g = ke^x. \quad (3.2)$$

In these coordinates Harish-Chandra's formula for the Haar measure of G is

$$dg = \prod_{\alpha > 0} \left| \frac{\sinh \alpha(\mathfrak{a}(x))}{\alpha(\mathfrak{a}(x))} \right|^2 dk \times dx \quad (3.3)$$

where $x \in \mathfrak{p}$ is K -conjugate to $\mathfrak{a}(x) \in \mathfrak{h}_{\mathbb{R}}$, and the product is over the positive roots (see [Helg], Thm 5.8, p 186).

Conjectural Step 7 of §2 is equivalent to

$$(g_0)_* \mu = \frac{1}{\mathcal{Z}} \frac{\sum_W (-1)^w \int \prod_{\alpha > 0} \langle \lambda, \alpha \rangle^2 \sinh(\pi \langle \lambda, \alpha \rangle / 2\dot{g})^{-1} a^{iw \cdot \lambda} d\lambda}{\prod_{\alpha > 0} (a^\alpha - a^{-\alpha})} dg_0. \quad (3.4)$$

where in this formula, for $g \in G$, $KgK = Ka(g)K$, $a \in \exp(\mathfrak{h}_{\mathbb{R}})/W$. This reduces to (1.12) for $K = SU_2$.

If $K = SU(2, \mathbb{C})$, then \mathfrak{p} consists of 2×2 Hermitian matrices, and we can use the standard identification

$$K \times \mathfrak{p} = S^3 \times (\mathbb{R}\vec{i} + \mathbb{R}\vec{j} + \mathbb{R}\vec{k}), \quad (3.5)$$

where $\vec{i} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\vec{j} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\vec{k} \leftrightarrow \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. In these coordinates

$$dg = \left(\frac{\sinh(2|x|)}{2|x|} \right)^2 dk \times dx, \quad (3.6)$$

where dk denotes Haar measure for $SU(2)$ and dx is Lebesgue measure for \mathbb{R}^3 . The formula (1.12) is then

$$\begin{aligned} (g_0)_* \mu_0 &= \frac{1}{\mathcal{Z}} \frac{1}{\cosh^3(2|x|)} \left(\frac{\sinh(2|x|)}{2|x|} \right)^2 dk \times dx \\ &= \frac{1}{\mathcal{Z}} \operatorname{sech}(2|x|) \left(\frac{\tanh(2|x|)}{2|x|} \right)^2 dk \times dx \\ &= \frac{1}{\mathcal{Z}} \delta(r) dr \times dk \times dA_{S^2}(x'), \end{aligned} \quad (3.7)$$

where $\delta = \operatorname{sech}(r) \tanh^2(r)$, $2x = rx'$, $x' \in S^2$ (see §4.4 of Part III of [Pi]).

Let $D = \frac{\partial}{\partial r}$, and let H_r denote the radial part of H_G . Since H_r is self-adjoint and nonnegative with respect to $\delta(r)dr$, and because H , hence H_r , applied to a constant (the vacuum) vanishes, H_r must necessarily be of the form

$$\begin{aligned} H_r^\alpha &= -\delta^{-1/2} \circ D \circ \alpha(r) \circ D \circ \delta^{1/2} + \frac{D(\alpha D \delta^{1/2})}{\delta^{1/2}} \\ &= \alpha[-\delta^{-1/2} \circ D^2 \circ \delta^{1/2} + \frac{D^2(\delta^{1/2})}{\delta^{1/2}}] - D(\alpha)D \end{aligned}$$

$$= \alpha[-\delta^{-1/2} \circ D^2 \circ \delta^{1/2} + \frac{1}{4} - \frac{15}{4} \text{sech}^2(r)] - D(\alpha)D \quad (3.8)$$

where $\alpha = \alpha(r)$ is a positive function. This is our initial ballpark conjecture.

The principal symbol of H_r^α , in the coordinate r , is $\alpha\xi^2$, where ξ is a variable dual to r . Determining α is thus equivalent to picking out a preferred geometry. We will now explain why $\alpha = 1$ appears to be a preferred choice.

We are assuming that H_r is the radial part of an operator H_G , and there is an intermediate operator $H_{\mathfrak{p}}$, acting on functions of \mathfrak{p} alone, in the Cartan decomposition (these are functions which are invariant with respect to the left action of K ; we could just as well consider the right action). The principal symbol of $H_{\mathfrak{p}}$ corresponds to a metric on \mathfrak{p} .

In considering interesting possibilities for the principal symbol of $H_{\mathfrak{p}}$, it seems that this metric has the form

$$g_x(v, w) = \langle A(ad(x))v, w \rangle, \quad (3.9)$$

where A is an analytic function which is expressible as a power series in powers of $ad(x)$, $x \in \mathfrak{p}$. In the appendix to this section, we will show that in all such cases, $\alpha = 1$ (see (f) of Lemma (A.2)).

Suppose that $\alpha = 1$. In this case H_r is equivalent to

$$-D^2 + \frac{1}{4} - \frac{15}{4} \text{sech}^2(r), \quad (3.10)$$

acting on odd functions of r . The restriction to odd functions of r is necessitated by the fact that $\delta^{1/2} = \text{sech}^{1/2}(r)\tanh(r)$ is an odd function (functions in the domain of H_r will then be of the form (odd function) $/\delta^{1/2}$, which will represent a well-defined function on G). This operator has a unique eigenvalue $\lambda = 0$ corresponding to the ground state, $\delta^{1/2}$, and the rest of the spectrum is continuous and of the form $[m, \infty)$, where $m = \frac{1}{4}$ is the mass gap (Note: if we remove the restriction on the domain of (3.10) to odd functions, then the operator has a lower energy state, the even function $\text{sech}^{3/2}(r)$, which corresponds to the eigenvalue -2). The scattering theory for the sech^2 potential (at least without domain restriction) is well-known (see e.g. §2.5 of [L]). Taking the domain restriction into account, this should be related to Zamolodchikov's conjectural S -matrix for this model (see [Z]).

In our argument for $\alpha = 1$, we noted that H_r does not determine the form of H_G (or $H_{\mathfrak{p}}$). At the level of G , we have

$$H_G = \Phi^{-1/2} \circ \Delta \circ \Phi^{1/2} - \frac{\Delta(\Phi^{1/2})}{\Phi^{1/2}}, \quad (3.11)$$

where Δ (a Laplace type operator) is self-adjoint with respect to $dk \times dx$ and $(g_0)_*\mu = \Phi(dk \times dx)$. There are numerous possibilities for Δ .

For example, relative to the Cartan decomposition $G = K \times \mathfrak{p}$, we could have $\Delta = \Delta^K + \Delta^{\mathfrak{p}}$, the sum of the Laplacians. For this example the $m = 1/4$ is directly related to the curvature of $G/K = H^3$ (relative to the normalization of our metric), because $\Delta^{G/K}$ is

equivalent to $\Delta^{\mathfrak{p}} + 1/4$ (see Proposition 3.10 of [Helg], pg 268, and (A.6) of the Appendix). This is relevant to the explanation for various miracles that occur in harmonic analysis in 3 versus n dimensions (see [Helg], especially p 266).

In the Appendix we consider a second possibility in detail. This second possibility is interesting because it generalizes to other Riemannian manifolds, in a way which seems linked to renormalization of sigma models.

We now discuss how to incorporate a level l , which presumably is related to the massive deformation of the conformally invariant WZW model at level $l > 0$.

We first recall from [Pi] that, at least conjecturally,

$$(g_0)_* \mu_l = \frac{1}{\mathcal{Z}} \chi_{l/2}(e^{2x}) \frac{1}{\cosh^3((2+l)|x|)} \left(\frac{\sinh(2|x|)}{2|x|} \right)^2 dk \times dx, \quad (3.12)$$

where $\chi_{l/2}$ is a character, at least for integer $l/2$ [μ_l denotes the measure gotten by coupling a certain density appropriate at level l ; it is the conjectural limit of the $\nu_{\beta,l}$ in (2.15); see [Pi]].

[“Proof of (3.12)”. In §4.4 of Part III of [Pi] we conjectured that (in the case $G = SL(2, \mathbb{C})$, $\Lambda = 0$, and λ is identified with $\lambda\alpha_1$, $\alpha_1 = \lambda_1 - \lambda_2$)

$$\int_{L_{hyp} G} \mathbf{a}(g)^{-i\lambda} d\mu_l = \frac{\sin(\frac{\pi}{2(2+l)} 2)}{\sin(\frac{\pi}{2(2+l)} 2(1 - i\lambda))} \quad (3.13)$$

Write $(g_0)_* \mu_l = \phi_l dm(g_0)$, where dm denotes G Haar measure. By (4.4.12) of [Pi]

$$\mathcal{H}\phi_l(\lambda) = \frac{i\lambda \sin(\frac{\pi}{2+l})}{\sin(\frac{\pi}{2+l} i\lambda)} = \sin(\frac{\pi}{2+l}) \frac{\lambda}{\sinh(\frac{\pi}{2+l} \lambda)} \quad (3.14)$$

By (4.4.15) of [Pi]

$$\begin{aligned} \phi_l\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) &= \frac{1}{\mathcal{Z}} \frac{1}{(a^{2+l} + a^{-(2+l)})^3} \frac{a^{2+l} - a^{-(2+l)}}{a^2 - a^{-2}} \\ &= \frac{1}{\mathcal{Z}} \cosh^{-3}((2+l)x) \chi_{l/2}\left(\begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix}\right), \end{aligned} \quad (3.15)$$

where $\chi_{l/2}$ is the character for the $SU(2)$ representation of dimension $l/2$ (assuming this is integral). This implies (3.12).]

Write $r = (2+l)|x|$, $a = \frac{2}{2+l}$, $D = \frac{\partial}{\partial r}$, and

$$\delta = \operatorname{sech}^3(r) \sinh(r) \sinh(ar), \quad (3.16)$$

so that the radial projection of $(g_0)_* \mu_l$ is (conjecturally) $\mathcal{Z}^{-1} \delta(r) dr$.

Assuming that $\alpha = 1$, for the same reasons cited above, we find that H_r^l is conjecturally of the form

$$\delta^{-1/2} \circ |D|^2 \circ \delta^{1/2} + \frac{1}{4}(5 + 2a^2 - 6a \frac{\tanh(r)}{\tanh(ar)} - (\operatorname{acoth}(ar) - \coth(r))^2 - 15 \operatorname{sech}^2(r)).$$

The potential well for this operator digs deeper as $l \rightarrow \infty$, suggesting that the number of eigenvalues and bound states goes to ∞ as $l \uparrow \infty$.

§4. The Finite R Case.

As we explained in the introduction, and in (1.5), for the sigma model with target K , we expect that there should be a natural Hilbert space $\mathcal{H}(S_R^1) = L^2(\frac{1}{\mathcal{Z}}\Omega_R^2 dV)$ for each $0 < R \leq \infty$, where heuristically we think of Ω_R as the vacuum state.

At this point we lack a construction, and a conjectural characterization (as in (1.9)), for the appropriate measure, when R is finite. However in this section we will assume this can be done. The point of this section is to explore what appears to be a natural conjecture for the g_0 distribution.

As in §3 we will write $g_0 = l_0 \mathbf{m} \mathbf{a} u_0$ for the triangular decomposition (when it exists), and $K g_0 K = K a K$ for the Cartan decomposition, where $\mathbf{a} \in A = \exp(\mathfrak{h}_{\mathbb{R}})$ and $a \in A/W$, respectively.

As in Chapter XXI of [WW], θ_1 will denote the odd theta function

$$\theta_1(x, \tau) = 2q^{1/8} \sin(x) - 2q^{3/8} \sin(3x) + \dots \quad (4.1)$$

$$= 2q^{1/8} \sin(x) \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e^{i2x})(1 - q^n e^{-i2x}), \quad (4.2)$$

where $q = \exp(2\pi i \tau)$ (this is the square of “ q ” in [WW]), $\operatorname{Im}(\tau) > 0$, and the equality is known as the Jacobi triple product formula. This theta function has the quasi-periodicity properties

$$\theta_1(x + \pi) = -\theta_1(x), \quad \theta_1(x + \tau) = -q^{1/2} e^{-2ix} \theta_1(x), \quad (4.3)$$

and zeroes at the points

$$x = n\pi + m\pi\tau, \quad m, n \in \mathbb{Z}. \quad (4.4)$$

Below we will also need to consider the even theta functions θ_3 and θ_4 , which have analogous properties (see [WW]).

(4.5) Conjecture. *The analogue of (2.15) (the diagonal distribution) is*

$$\int \mathbf{a}^{-i\lambda} \frac{1}{\mathcal{Z}} \Omega_{R,l}^2 dV = c \prod_{\alpha > 0} \left\{ \frac{\sinh(\frac{\pi}{2R(\dot{g}+l)} \langle \rho - i\lambda, \alpha \rangle)}{\langle \rho - i\lambda, \alpha \rangle \theta_1(\frac{\pi}{2(\dot{g}+l)} \langle \rho - i\lambda, \alpha \rangle, iR)} \right\} \quad (4.6)$$

where c is determined by the condition that the right hand side of (4.6) is 1 at $\lambda = 0$.

If we write $(g_0)_*(\frac{1}{Z}\Omega_{R,l}^2 dV) = \phi_{R,l}(g_0)dm(g_0)$, where dm denotes Haar measure for G , then (4.5) is equivalent to

$$(\mathcal{H}\phi_{R,l})(\lambda) = c \prod_{\alpha > 0} \left\{ \frac{\sin(\frac{\pi}{2R(\dot{g}+l)} \langle \lambda, \alpha \rangle)}{\theta_1(\frac{\pi}{2(\dot{g}+l)} \langle i\lambda, \alpha \rangle, iR)} \right\} \quad (4.7)$$

for the Harish-Chandra transform. Note that the zeros of the sine function in (4.7) exactly cancel with the zeros of $\theta_1(i\cdot)$, so the α factor in (4.7) is smooth and rapidly decreasing as a function of the single variable $\langle \lambda, \alpha \rangle$, for each positive root α .

The motivations for this conjecture are rather vague: the philosophy that theta functions are natural q -deformations of trigonometric functions, the relevance of the q -deformation of the affine algebra $\hat{L}\mathfrak{g}$ to integrable models (see [Sm] and references there), and the surprising appearance of “ τ ” in similar models (especially gauge theories; see e.g. [Wi2]).

To show that this formula is reasonable, there are several things that need to be checked. The first is to note that (4.6) reduces to (2.15) when $R \uparrow \infty$. This follows in an elementary way from (4.1) (in verifying this, one must bear in mind the dependence of c on R). Thus this formula is consistent with our earlier claim.

Secondly we need to know that the transforms we are writing down actually correspond to positive measures. We first consider (4.6).

(4.8)Proposition. *The right hand side of (4.6) is a positive definite function of $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$.*

Proof of (4.8). Products of positive definite functions are positive definite. In (4.6) we have a product over roots, and it suffices to show that each factor is positive definite as a function of one variable (the distance from $\ker(\langle \cdot, \alpha \rangle)$).

The dual Coxeter number is given by $\dot{g} = 1 + \langle \rho, \theta \rangle / 2$, where θ is the highest root and (we recall that) ρ is the sum of the positive roots. Using this and the fact that $\langle \rho, \alpha \rangle \leq \langle \rho, \theta \rangle$, for each root α , we can write

$$\frac{\pi}{2(\dot{g}+l)} (\langle \rho, \alpha \rangle - i\langle \lambda, \alpha \rangle) = x_0 + iy, \quad (4.9)$$

where $0 < x_0 < \pi$, and y is a scaling of the variable $\langle \lambda, \alpha \rangle$. Since scaling a positive definite function does not change its positivity, it therefore suffices to prove that

$$\frac{\sinh((x_0 + iy)/R)}{\theta_1(x_0 + iy, iR)(x_0 + iy)/R} \quad (4.10)$$

is a positive definite function of y .

This is a consequence of the following striking result, which probably is known.

(4.11)Lemma. For $0 < x_0 < \pi$,

$$\frac{1}{2\pi} \int \frac{1}{\theta_1(x_0 + iy, iR)} e^{ipy} dy = \frac{1}{\theta'_1(0, iR)} \frac{\theta_4(\pi R p/2, iR)}{e^{x_0 p} + e^{(x_0 - \pi)p}}. \quad (4.12)$$

Proof of (4.11). This is a straightforward residue calculation. Suppose that $p > 0$. The residues for the integrand on the LHS of (4.12), as a function of complex y , occur at the points $y = m\pi R + i(x_0 + n\pi)$, $n, m \in \mathbb{Z}$, $n \geq 0$. Thus the LHS of (4.12) equals

$$i \sum_{m \in \mathbb{Z}, n \geq 0} \frac{\exp(ip(m\pi R + i(x_0 + n\pi)))}{i\theta'_1(-n\pi + im\pi R)} \quad (4.13)$$

Using the quasi-periodicity properties of θ_1 , (4.3), we obtain

$$\theta'_1(-n\pi + im\pi R) = (-1)^{n+m} q^{-m^2/2} \theta'_1(0). \quad (4.14)$$

Thus (4.13) equals

$$\frac{e^{-x_0 p}}{\theta'_1(0)} \left(\sum_{n=0}^{\infty} (-1)^n e^{-\pi p n} \right) \left(\sum_{m \in \mathbb{Z}} (-1)^m q^{m^2/2} e^{iR p m} \right). \quad (4.15)$$

The second sum is expressible in terms of θ_4 , and this implies (4.11). //

The Fourier transform of $\sin(y)/y$ is essentially a characteristic function. This together with the Lemma implies that the inverse Fourier transform of (4.10), as a function of p , is the convolution of measures

$$\frac{1}{\theta'_1(0, iR)} \frac{1}{e^{x_0 p} + e^{(x_0 - \pi)p}} \theta_4(\pi R p/2, iR) * e^{-x_0 p} \chi_{[-1/R, 1/R]}(p)/2R \quad (4.16)$$

The crucial fact now is that the function $\theta_4(x, iR)$ is positive for $x \in \mathbb{R}$. Thus both measures are positive, implying that the convolution is positive. This completes the proof of (4.8). //

(4.17)Remark. Another possible approach to (4.11) is to consider the Jacobi triple product formula for θ_1 , (4.2), which corresponds to an (infinite) convolution product formula for (4.11). If we compute the inverse Fourier transform for the n th term, we find that for $0 < x_0 < \pi$,

$$\frac{1}{2\pi} \int \frac{1}{1 - 2\cos(2(x_0 + iy))q^n + q^{2n}} e^{ipy} dy \quad (4.18)$$

$$= \frac{\sin(\pi n R p)}{\sinh(\pi n R)(e^{x_0 p} + e^{(x_0 - \pi)p})}. \quad (4.19)$$

which is highly oscillatory. From this point of view the positivity of (4.11) is surprising.

We now consider the formula (4.7) for the Harish-Chandra transform. The abstract inversion formula is

$$\phi_{R,l}(g_0) = \frac{1}{\mathcal{Z} \prod_{\alpha>0} (a^\alpha - a^{-\alpha})} \sum_W (-1)^w \int \prod_{\alpha>0} \{ \langle \lambda, \alpha \rangle^2 \frac{\sin(\frac{\pi}{2R(\dot{g}+l)} \langle \lambda, \alpha \rangle)}{\theta_1(\frac{\pi}{2(\dot{g}+l)} \langle \lambda, \alpha \rangle)} \} a^{iw \cdot \lambda} d\lambda \quad (4.20)$$

One can change variables in the integrals to reduce the calculations to the case $l = 0$.

We will analyze this in the case $K = SU(2)$.

(4.21) Lemma.

$$\frac{i}{2\pi} \int \frac{y}{\theta_1(iy)} \frac{\sinh(iy/R)}{iy/R} e^{ipy} dy = \frac{R \sinh(\pi/R) \theta_3(\pi R p/2, iR)}{4\theta'_1(0)(\sinh^2(\pi/(2R)) + \cosh^2(\pi p/2))}. \quad (4.22)$$

Proof of (4.21). This is another straightforward residue calculation. Suppose that $p > 0$. As a function of the complex variable y , the singularities of the integrand on the LHS of (4.22) occur at the points $y = m\pi R + in\pi$, $m, n \in \mathbb{Z}$, $n > 0$. Using the formula (4.14) to calculate the residues, we see that the LHS of (4.22) equals

$$= R \sum_{n>0, m} \frac{\exp(-\pi p n + i\pi R p m) \sinh((-n\pi + imR\pi)/R)}{(-1)^n (-1)^m q^{-m^2/2} \theta'_1(0)} \quad (4.23)$$

$$= \frac{R}{\theta'_1(0)} \left(\sum_m q^{m^2/2} e^{ipRm} \right) \left(\sum_{n>0} (-1)^n \sinh(-n\pi/R) e^{-n\pi p} \right) \quad (4.24)$$

$$= \frac{R}{2\theta'_1(0)} \theta_3(\pi R p/2) \left\{ \frac{e^{-\pi/R}}{e^{\pi p} + e^{-\pi/R}} - \frac{e^{\pi/R}}{e^{\pi p} + e^{\pi/R}} \right\} \quad (4.25)$$

After some elementary manipulations, this leads to (4.22). //

For the $SU(2, \mathbb{C})$ case we employ the same notation as in §3. Thus we identify $a \in \exp(\mathfrak{h}_{\mathbb{R}})$ with $\begin{pmatrix} e^x & \\ & e^{-x} \end{pmatrix}$, $x \in \mathbb{R}$, we identify $\lambda \in \mathfrak{h}_{\mathbb{R}}$ with $\lambda \alpha_1$, where $\lambda \in \mathbb{R}$ and α_1 is the positive root for $sl(2, \mathbb{C})$, and we write $r = 2|x|$. We then have

$$\phi_{R,l}(g) = \frac{1}{\mathcal{Z}} \frac{1}{\sinh(2|x|)} \left(-\frac{\partial}{\partial z} \left\{ \frac{\theta_3(Rz, iR)}{\sinh(\pi/(2R))^2 + \cosh(z)^2} \right\} \right)_{|z=(2+l)x/\pi}. \quad (4.26)$$

where \mathcal{Z} is a normalization constant, so that the integral with respect to Haar measure for $SL(2, \mathbb{C})$ is one.

(4.27) Proposition. *We have $\phi_{R,l} \geq 0$.*

Proof of (4.27). By doing the differentiation in (4.26), we see that (4.27) is equivalent to

$$\frac{\partial}{\partial z} \ln(\theta_3(Rz, iR)) \leq 2 \frac{\sinh(z) \cosh(z)}{\sinh(\pi/(2R))^2 + \cosh(z)^2}, \quad z > 0. \quad (4.28)$$

The LHS of (4.28) has period π/R . The function $\theta_3(Rz, iR)$ is decreasing on $[0, \pi/(2R)]$, so the LHS of (4.28) is negative on this interval, and hence the claim is trivially true on this interval. It is straightforward to check that the RHS of (4.28) is an increasing function of z . Thus it suffices to prove (4.28) on the finite interval $[\pi/(2R), \pi/R]$ (Note this means that for values of R on the order of 1, one can with confidence simply look at the graph of $\theta_3(Rz, iR)/(\sinh(\pi/(2R))^2 + \cosh(z)^2)$, and check that it is decreasing on the appropriate interval).

The LHS of (4.28) equals

$$-4R \sin(2Rz) \sum_{n=1}^{\infty} \frac{q^{n-1/2}}{1 + 2\cos(2Rz)q^n + q^{2n}}. \quad (4.29)$$

The maximum of this function of $\pi/(2R) \leq z \leq \pi/R$ is the same as the maximum of the function

$$2R \sin(\theta) e^{\pi R} \sum_{n=1}^{\infty} \frac{1}{\cosh(2\pi Rn) - \cos(\theta)}, \quad 0 \leq \theta \leq \pi, \quad (4.30)$$

where $z = (2\pi - \theta)/(2R)$.

We first derive an easy bound for (4.30), which is sufficient for R sufficiently large. On the domain $0 \leq \theta \leq \pi$, the function $\sin(\theta)/(\cosh(2\pi Rn) - \cos(\theta))$ has a maximum value of $1/\sinh(2\pi Rn)$, which is achieved at the point $\theta = \theta_{R,n}$ satisfying $\cos(\theta) = \cosh(2\pi Rn)^{-1}$. Thus (4.30) is bounded by

$$2Re^{\pi R} \sum_{n=1}^{\infty} \frac{1}{\sinh(2\pi Rn)}, \quad (4.31)$$

which is a decreasing function of R . It is easy to check that this is dominated by the minimum of the RHS of (4.28),

$$\frac{2\sinh(\pi/(2R))\cosh(\pi/(2R))}{\sinh(\pi/(2R))^2 + \cosh(\pi/(2R))^2}, \quad (4.32)$$

for R sufficiently large (in fact for $R > 1/20$ (using Maple, for example)). But (4.31) diverges as $R \downarrow 0$, and so this does not work in general.

Now consider small R . The function (4.30) vanishes at 0 and π , and it has a unique maximum at a point θ_R in the interior. This point is determined by setting the derivative of (4.30) to zero, and this gives rise to the equation

$$\sum_{n=1}^{\infty} \frac{\cos(\theta_R) \cosh(2\pi Rn) - 1}{(\cosh(2\pi Rn) - \cos(\theta_R))^2} = 0, \quad (4.33)$$

which is not solvable. However, we previously calculated the unique critical points for the terms in (4.30), and from this we see that

$$\theta_R \geq \min\{\theta_{R,n} : n \geq 1\} = 2\pi R. \quad (4.34)$$

This will allow us to avoid multiple cases below.

Since $\cosh(x) \geq 1 + x^2/2$, the function (4.30) is bounded by

$$2Re^{\pi R} \sin(\theta) \sum_{n=1}^{\infty} \frac{1}{(1 - \cos(\theta)) + (2\pi Rn)^2}. \quad (4.35)$$

The Poisson summation formula (applied to the function $f(x) = \exp(-|x|)$) implies the identity

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^2 + n^2} = \frac{1}{2} \left(\frac{\pi}{\alpha} \coth(\pi\alpha) - \frac{1}{\alpha^2} \right). \quad (4.36)$$

This identity, with $\alpha^2 = (1 - \cos(\theta))/(2\pi R)$, implies that (4.35) equals

$$Re^{\pi R} \sin(\theta) \frac{1}{(2\pi R)^2} \left(\frac{\pi \cdot 2\pi R}{(1 - \cos(\theta))^{1/2}} \coth(\pi\alpha) - \frac{(2\pi R)^2}{1 - \cos(\theta)} \right) \quad (4.37)$$

Thus (4.30) is bounded by

$$e^{\pi R} \frac{\sin(\theta)}{2(1 - \cos(\theta))^{1/2}} \coth(\pi\alpha). \quad (4.38)$$

For sufficiently small R , because of (4.34),

$$1 - \cos(\theta_R) \geq \frac{1}{2}(2\pi R)^2. \quad (4.39)$$

This implies that $\alpha(\theta_R) \geq 1/\sqrt{2}$. Because \coth is decreasing, and $\sin(\theta)(1 - \cos(\theta))^{-1/2}$ is bounded by $\sqrt{2}$, this implies that (4.38) is bounded by

$$e^{\pi R} 2^{-1/2} \coth(\pi/\sqrt{2}) \leq (.73)e^{\pi R}. \quad (4.40)$$

This is bounded by (4.32) (which is very close to 1), for $R < 1/10$ (using Maple). //

We now specialize to the case $l = 0$. In the analogue of (3.7) we have

$$\delta = \delta_R(r) = \frac{1}{\mathcal{Z}} \left(-\frac{\partial}{\partial r} \left\{ \frac{\theta_3(Rr, iR)}{\sinh(\pi/2R)^2 + \cosh(r)^2} \right\} \right) \sinh(r). \quad (4.41)$$

This is positive, and its square root is the vacuum. The corresponding potential function is given by

$$q_R(r) = \frac{D^2(\delta^{1/2})}{\delta^{1/2}} = \frac{1}{2} \left[\frac{\delta''}{\delta} - \frac{1}{2} \left(\frac{\delta'}{\delta} \right)^2 \right] = \frac{1}{2} [(\ln \delta)'' + \frac{1}{2} (\ln \delta)'^2]. \quad (4.42)$$

When $R = \infty$, this is $\frac{1}{4} - \frac{15}{4} \text{sech}^2(r)$, which is bounded. As we explained in the introduction, we would like to believe that the operator $-D^2 + q_R$ has discrete spectrum, when $R < \infty$. This is equivalent to showing that q_R is unbounded as $r \uparrow \infty$, for $R < \infty$. Unfortunately I have not been able to resolve this issue.

Appendix.

We identify \mathfrak{p} with \mathbb{R}^n (with $n = 3$ in our rank one case), as in (3.5) above, so that our preferred inner product on \mathfrak{p} is twice the Euclidean dot product. Now suppose that we are given a metric on \mathbb{R}^n ,

$$g_x(\xi, \eta) = A(x)\xi \cdot \eta, \quad (A.1)$$

where $A(x)$ is a positive matrix for each $x \in \mathbb{R}^n$. Let ∇ , dV , .. denote the usual Euclidean gradient, volume element, ...

(A.2) Lemma. *We have*

$$(a) \nabla^g = A^{-1} \nabla$$

$$(b) dV_g = \rho dV, \rho = \det(A)^{1/2}$$

$$(c) \text{div}^g(v) = \rho^{-1} \text{div}(\rho v), v \in \text{Vect}(\mathbb{R}^n)$$

$$(d) \Delta^g = -\rho^{-1} \text{div}(\rho A^{-1} \nabla(\cdot))$$

$$(e) Q_g(f) = \int (\Delta^g f) \bar{f} \rho dV = \int g(A^{-1} \nabla f, A^{-1} \nabla f) \rho dV.$$

Assuming that g is orthogonally invariant, we also have

$$(f) \text{ For } f = f(r),$$

$$Q_g(f) = \int_0^\infty f'^2 \alpha(r) \rho(r) r^{n-1} dr, \quad (A.3)$$

$$\Delta_r^g = -\delta^{-1/2} \circ D \circ \alpha \circ D \circ \delta^{1/2} + \frac{1}{2} \left[\frac{(\alpha \delta')'}{\delta} - (\ln \delta)'^2 \right], \quad (A.4)$$

where $\delta(r) = \rho(r)r^{n-1}$ and $\alpha(r) = A^{-1}(x) \frac{x}{r} \cdot \frac{x}{r}$, for any choice of x with $|x| = r$. In particular, if A is analytic and locally expressible as a power series in $ad(x)$, with $A(0) = 1$, then $\alpha = 1$.

Proof of (A.2). Parts (a)-(e) are routine, and the formula for Q_g in (A.3) follows directly from (e). For (A.4), since Δ_r^g is self-adjoint with respect to $\delta(r)dr$, we know a priori that Δ_r^g has the form

$$-\delta^{-1/2} \circ D \circ \alpha \circ D \circ \delta^{1/2} + \gamma. \quad (A.5)$$

We can plug this form into Q_g in (e) and compare with (A.3). This determines α and γ . //

(A.6)Example. Suppose first that we identify $\mathfrak{p} \rightarrow G/K : x \rightarrow e^x K$, where the latter space is equipped with its negatively curved metric. We have

$$\begin{aligned} g_x^{G/K}(\xi, \eta) &= \left\langle \frac{d}{dt} \Big|_{t=0} e^{x+t\xi}, \dots \right\rangle_{e^x} \\ &= A(ad(x))\xi \cdot \eta, \end{aligned} \quad (\text{A.7})$$

where $A(x) = \left| \frac{1-e^{-z}}{z} \Big|_{z=ad(x)} \right|^2$ (using a standard formula for derivative of the exponential map). In the case $G = SL(2, \mathbb{C})$, so that $\mathfrak{p} = \mathbb{R}^3$,

$$\begin{aligned} \rho(r) &= \det\left(\frac{1-e^{-z}}{z} \Big|_{z=ad(x)}\right) = \frac{1-e^{-r}}{r} \cdot \frac{1-e^r}{r} \\ &= \frac{2-2\cosh(r)}{r^2} = \frac{\sinh^2(\frac{r}{2})}{(\frac{r}{2})^2} \end{aligned} \quad (\text{A.8})$$

which is consistent with Harish-Chandra's formula, if we remember how things are normalized. Now (f) implies that

$$\Delta^{G/K} = -\delta^{1/2} \circ D^2 \circ \Delta^{1/2} + \frac{1}{4}. \quad (\text{A.9})$$

(A.10)Example. We consider the Guillemin-Stenzel Kahler structure for $G = K \times \mathfrak{p} = K \times \mathfrak{k} = TK$, where $\mathfrak{k} \rightarrow \mathfrak{p} : v \rightarrow iv$ ([St]). This is interesting to consider, because there is a conjectural generalization of this to a general compact Riemannian manifold X with $Ric \geq 0$ (a condition related to the renormalizability of sigma models).

The complex structure is the usual one for G . If we identify the tangent space of $K \times \mathfrak{p}$ with $\mathfrak{k} \otimes \mathfrak{p}$ using left translation, then at the point $(k, x) \in K \times \mathfrak{p}$, the complex structure is given by

$$J_{(k,x)}\left(\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) = i \begin{pmatrix} \frac{1-\cosh(z)}{\sinh(z)} & \frac{2(\cosh(z)-1)}{z\sinh(z)} \\ \frac{z}{\sinh(z)} & \frac{\cosh(z)-1}{\sinh(z)} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \Big|_{z=ad(x)} \quad (\text{A.11})$$

where “ i ” stands for usual multiplication by i on $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

The canonical T^*K symplectic structure, in the Cartan coordinates $K \times \mathfrak{p}$, is constant and given by

$$\omega\left(\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi' \\ \eta' \end{pmatrix}\right) = \langle i\xi \otimes \eta' - \eta \otimes i\xi' \rangle; \quad (\text{A.12})$$

note that $i\xi \in \mathfrak{p}$, so that the inner product makes sense.

It follows from these calculations that the Riemannian metric is given by

$$g_{(k,x)}\left(\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) = \omega\left(J_{(k,x)}\begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right)$$

$$\begin{aligned}
&= \omega\left(i \begin{pmatrix} \frac{1-\cosh(z)}{\sinh(z)} & \frac{2(\cosh(z)-1)}{z\sinh(z)} \\ \frac{z}{\sinh(z)} & \frac{\cosh(z)-1}{\sinh(z)} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \Big|_{z=ad(x)}, \begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) \\
&= \langle \left(\frac{1-\cosh(z)}{\sinh(z)}\xi + \frac{2(\cosh(z)-1)}{\sinh(z)}\eta \right) \otimes \eta - \left(\frac{z}{\sinh(z)}\xi + \frac{\cosh(z)-1}{\sinh(z)}\eta \right) \otimes \xi \rangle \Big|_{z=ad(x)} \\
&= \langle \frac{ad(x)}{\sinh(ad(x))}\xi \otimes \xi + 2\frac{\cosh(ad(x))-1}{ad(x)\sinh(ad(x))}\eta \otimes \eta \rangle
\end{aligned} \tag{A.13}$$

where the bracket denotes the negative of the (appropriate multiple of) the Killing form.

Now suppose that we just consider \mathfrak{p} . In this case

$$A(ad(x)) = 2\frac{\cosh(z)-1}{z\sinh(z)} \Big|_{z=adx} = \frac{\tanh(w)}{w} \Big|_{w=\frac{1}{2}adx}. \tag{A.14}$$

and $\rho(r) = 2\tanh(r/2)/r$. Again, $\alpha = 1$.

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